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STRESS INTENSITY FACTORS FOR SLIGHTLY KINKED, PARTIALLY CLOSED CRACKS IN COMPRESSION

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Abstract—A solution is presented for the elastic stress intensity factors at the tips of a slightly kinked, partially closed crack in compression. The solution is accurate to first order in the deviation of the crack surface from a straight line and is carried out using perturbation procedures analogous to those of Banichuk (*Izv. Akad. Nauk SSSR Mekh. Tverd. Tela* 7, 130 (1970)), Goldstein and Salganik (*Int. J. Fracture* **10**, 507 (1974)) and Cotterell and Rice (*Int. J. Fracture* **16**, 155 (1980)) for the problem of an open crack. Comparison with the exact solution indicates that the asymptotic solution is accurate for values of the angle between the straight crack and its out-of-plane kinks up to about 20°.

1. INTRODUCTION

Experiments on glass plates containing pre-existing planar through cracks oriented at an angle to the direction of the axial compression have revealed that the relative sliding of the faces of the pre-existing cracks does not result in co-planar crack growth, but rather produces at the tips of the pre-existing cracks small tension cracks which deviate at sharp angles from the sliding plane[1–4]. These experiments are designed to be models for the propagation of cracks in rocks in compression. In this paper, we are concerned with the calculation of stress intensity factors at the tips of the kinked open extensions of a closed sliding through crack. The same method can be extended to a curved crack with several closed sections. The solution obtained is accurate to first order in the deviation of the crack surface from a straight line drawn between the kink tips and is carried out using perturbation procedures similar to those used in Refs [5–9] for the problem of the open crack. The results can be stated in terms of known solutions for a single straight crack or a co-linear array of straight cracks.

A complete solution to the problem of the sliding kinked crack has been given by Nemat-Nasser and Horii[3], who used a continuous distribution of dislocations to model the crack and its kinks. In order to find the stress intensity factors, they solved numerically a singular integral equation for the dislocation distribution. In contrast, we can avoid the solution of the singular integral equation by using the results of the asymptotic analysis for the stress intensity factors. However, the validity of the asymptotic solution is limited to small deviations of the crack surface from a straight line. Comparisons with the exact solution given in Ref. [3] indicate that the first-order solution for the mode I stress intensity factor is accurate for values of the angle between the straight crack and its out-of-plane kinks up to about 20°.

2. GENERAL FORMULATION OF THE PROBLEM

2.1. *Formulation of the boundary value problem*

Consider an infinite plate of a homogeneous, isotropic, linearly elastic, brittle solid containing a curved crack on $y = \lambda(x)$, with its tips at positions $x = \pm a$ (Fig. 1). A uniform state of stress σ_{xx}^{∞}, σ_{yy}^{∞} and σ_{xy}^{∞} is applied at infinity, with $\sigma_{yy}^{\infty} < 0$ and $\sigma_{xy}^{\infty} < 0$, where tension is

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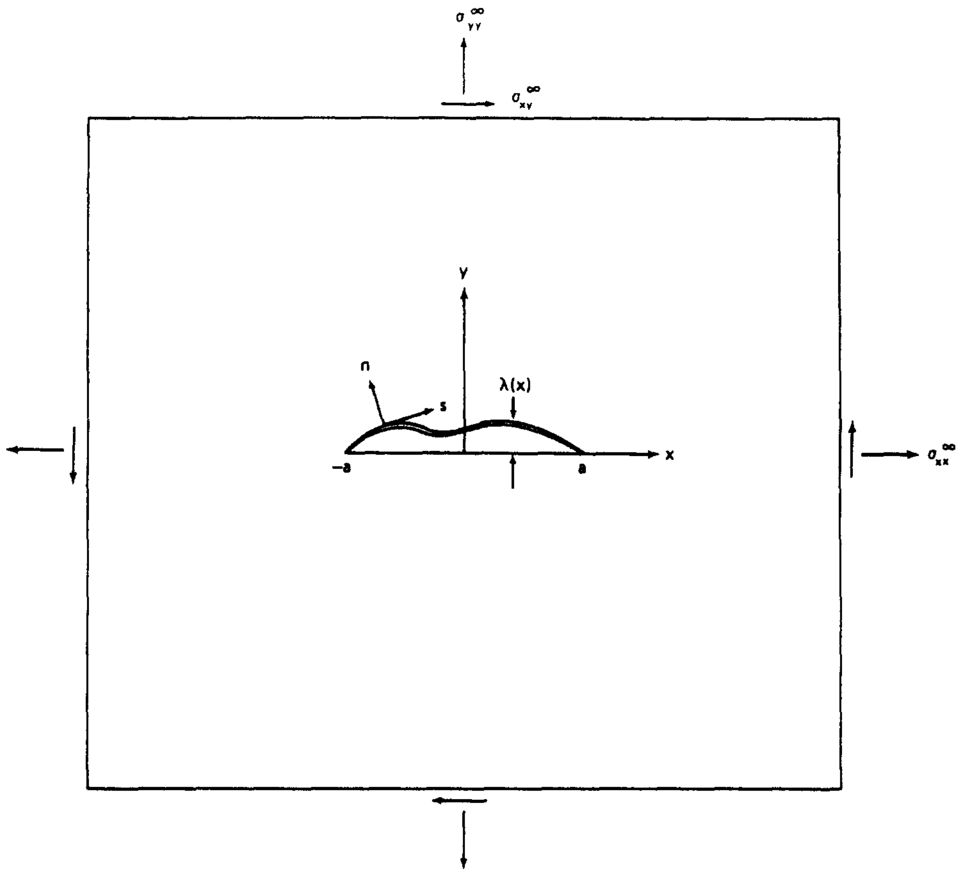

Fig. 1. Infinite plate containing a curved crack.

regarded as positive. The corresponding two-dimensional boundary value problem is given by

$$\left.\begin{aligned} \sigma_{ji,j} &= 0 \\ 2\varepsilon_{ij} &= u_{i,j}+u_{j,i} \\ \sigma_{ij} &= C_{ijkl}\varepsilon_{kl} \end{aligned}\right\} \quad \text{in } V \tag{1}$$

$$\sigma_{ij} = \sigma_{ij}^{\infty} \quad \text{at infinity} \tag{2}$$

$$\sigma_{nn}(x,\lambda) = \sigma_{ns}(x,\lambda) = 0 \quad \text{on the open portions of the crack} \tag{3}$$

$$\left.\begin{aligned} \sigma_{ns}(x,\lambda) &= \mu\sigma_{nn}(x,\lambda) \\ u_n^+(x,\lambda) &= u_n^-(x,\lambda) \end{aligned}\right\} \quad \text{on the sliding portions of the crack} \tag{4}$$

where σ_{ij}, ε_{ij} and u_i are the stress, strain and displacement fields in the region V occupied by the body, C_{ijkl} is the fourth-order tensor of the elastic moduli, σ_{nn} and σ_{ns} are the normal and shear tractions at the crack surface, u_n is the displacement in the direction normal to the crack surface, μ is the coefficient of friction, $A_{,i}$ is $\partial A/\partial x_i$ and the superscripts plus and minus denote the value of the indicated quantity on the upper and lower surfaces of the crack. Note that the open and sliding portions of the crack are, in general, not known in advance and their determination becomes part of the solution.

2.2. *Small-parameter expansion*

The essence of the approximation we use is that the solution to the problem with the curved crack is close, in some sense, to the solution of a similar problem for a straight crack. In fact, we shall use the solution to the following problem, involving a flat crack, as the leading or zero-order approximation in our expansion. Let $\boldsymbol{\sigma}^{(0)}$, $\boldsymbol{\varepsilon}^{(0)}$ and $\mathbf{u}^{(0)}$ be such that

$$\left.\begin{aligned} \sigma^{(0)}_{ji,j} &= 0 \\ 2\varepsilon^{(0)}_{ij} &= u^{(0)}_{i,j} + u^{(0)}_{j,i} \\ \sigma^{(0)}_{ij} &= C_{ijkl}\varepsilon^{(0)}_{kl} \end{aligned}\right\} \text{ in } V' \tag{5}$$

and

$$\sigma^{(0)}_{ij} = \sigma^{\infty}_{ij} \quad \text{at infinity} \tag{6}$$

$$\sigma^{(0)}_{yy}(x,0) = \sigma^{(0)}_{xy}(x,0) = 0 \quad \text{on the open portions of the crack} \tag{7}$$

$$\left.\begin{aligned} \sigma^{(0)}_{xy}(x,0) &= \mu\sigma^{(0)}_{yy}(x,0) \\ u^{+}_{y}(x,0) &= u^{-}_{y}(x,0) \end{aligned}\right\} \begin{array}{l}\text{on the sliding portions}\\ \text{of the crack}\end{array} \tag{8}$$

where V′ is the plane with a straight slit lying on the x-axis from $-a$ to a. If the slope of the actual crack, $\lambda'(x)$, has order of magnitude $\varepsilon \ll 1$ at its largest, then we can seek a perturbation expansion in ε for the solution to the problem of the curved crack, such that

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(0)} + \boldsymbol{\sigma}^{(1)} + O(\varepsilon^2) \tag{9}$$

$$\boldsymbol{\sigma} = \boldsymbol{\varepsilon}^{(0)} + \boldsymbol{\varepsilon}^{(1)} + O(\varepsilon^2) \tag{10}$$

$$\mathbf{u} = \mathbf{u}^{(0)} + \mathbf{u}^{(1)} + O(\varepsilon^2) \tag{11}$$

where $\boldsymbol{\sigma}^{(1)}$, $\boldsymbol{\varepsilon}^{(1)}$ and $\mathbf{u}^{(1)}$ are all $O(\varepsilon)$ compared to the leading order terms. We mention that $\lambda'(x) = O(\varepsilon)$ also means that $\lambda(x)/a = O(\varepsilon)$, because $\lambda(\pm a) = 0$. What remains now is the finding of the equations and the boundary conditions governing $\boldsymbol{\sigma}^{(1)}$, $\boldsymbol{\varepsilon}^{(1)}$ and $\mathbf{u}^{(1)}$. We note at this stage that our approach is identical to that of Cotterell and Rice[7], except that they addressed the problem of a crack open everywhere. Furthermore, they found their solutions and expressed their expansions in terms of Muskhelishvili's[10] complex potentials. We prefer to work in terms of fundamental quantities, although it is entirely possible that the partially closed, slightly curved crack can also be solved by a variation of the complex variable treatment of Cotterell and Rice[7].

We return now to the question of finding $\boldsymbol{\sigma}^{(1)}$, $\boldsymbol{\varepsilon}^{(1)}$ and $\mathbf{u}^{(1)}$. In order to find the equations and boundary conditions governing $\boldsymbol{\sigma}^{(1)}$, $\boldsymbol{\varepsilon}^{(1)}$ and $\mathbf{u}^{(1)}$, we substitute the expansions (9)–(11) into eqns (1)–(4). We also use the fact that both $\lambda(x)$ and $\lambda'(x)$ are $O(\varepsilon)$ to write expansions in ε for the tractions and displacements on the crack surface $y = \lambda(x)$. Using a tensorial transformation, we find that the normal and shear tractions on the actual crack can be written as

$$\begin{aligned} \sigma_{nn}(x,\lambda) = {} & \tfrac{1}{2}[\sigma_{xx}(x,\lambda) + \sigma_{yy}(x,\lambda)] \\ & + \tfrac{1}{2}[\sigma_{yy}(x,\lambda) - \sigma_{xx}(x,\lambda)]\cos 2\theta - \sigma_{xy}(x,\lambda)\sin 2\theta \end{aligned}$$

$$\sigma_{ns}(x,\lambda) = \sigma_{xy}(x,\lambda)\cos 2\theta + \tfrac{1}{2}[\sigma_{yy}(x,\lambda) - \sigma_{xx}(x,\lambda)]\sin 2\theta$$

where $\theta = \lambda'(x) + O(\varepsilon^3)$. Then, using a Maclaurin series expansion in θ for $\sin 2\theta$ and $\cos 2\theta$, we find

$$\sigma_{nn}(x,\lambda) = \sigma_{yy}(x,\lambda) - 2\lambda'(x)\sigma_{xy}(x,\lambda) + O(\varepsilon^2)$$

$$\sigma_{ns}(x,\lambda) = \sigma_{xy}(x,\lambda) + \lambda'(x)[\sigma_{yy}(x,\lambda) - \sigma_{xx}(x,\lambda)] + O(\varepsilon^2).$$

If we now write Maclaurin series expansions in y for σ_{xx}, σ_{yy} and σ_{xy}, the last two equations become

$$\sigma_{nn}(x,\lambda) = \sigma_{yy}(x,0) - \lambda(x)\frac{\partial\sigma_{xy}(x,0)}{\partial x} - 2\lambda'(x)\sigma_{xy}(x,0) + O(\varepsilon^2) \tag{12}$$

$$\sigma_{ns}(x,\lambda) = \sigma_{xy}(x,0) - \lambda(x)\frac{\partial\sigma_{xx}(x,0)}{\partial x} + \lambda'(x)[\sigma_{yy}(x,0) - \sigma_{xx}(x,0)] + O(\varepsilon^2) \tag{13}$$

where we have also used the equilibrium equations $\partial\sigma_{xy}/\partial y = -\partial\sigma_{xx}/\partial x$ and $\partial\sigma_{yy}/\partial y = -\partial\sigma_{xy}/\partial x$.

In a similar way we can show that

$$u_n(x,\lambda) = u_y(x,0) + \lambda(x)\varepsilon_{yy}(x,0) - \lambda'(x)u_x(x,0) + O(\varepsilon^2). \tag{14}$$

Using the expansions (9)–(11), eqns (12)–(14) can be written as

$$\sigma_{nn}(x,\lambda) = \sigma_{yy}^{(0)}(x,0) + \sigma_{yy}^{(1)}(x,0) - \lambda(x)\frac{\partial\sigma_{xy}^{(0)}(x,0)}{\partial x} - 2\lambda'(x)\sigma_{xy}^{(0)}(x,0) + O(\varepsilon^2) \tag{15}$$

$$\sigma_{ns}(x,\lambda) = \sigma_{xy}^{(0)}(x,0) + \sigma_{xy}^{(1)}(x,0) - \lambda(x)\frac{\partial\sigma_{xx}^{(0)}(x,0)}{\partial x} + \lambda'(x)[\sigma_{yy}^{(0)}(x,0) - \sigma_{xx}^{(0)}(x,0)] + O(\varepsilon^2) \tag{16}$$

$$u_n(x,\lambda) = u_y^{(0)}(x,0) + u_y^{(1)}(x,0) + \lambda(x)\varepsilon_{yy}^{(0)}(x,0) - \lambda'(x)u_x^{(0)}(x,0) + O(\varepsilon^2). \tag{17}$$

Finally, substituting eqns (9)–(11) and (15) and (16) into the boundary value problem formulated in Section 3.1 (eqns (1)–(4)), taking into account (5)–(8) and separating zero- and first-order terms, we find that $\boldsymbol{\sigma}^{(1)}$, $\boldsymbol{\varepsilon}^{(1)}$ and $\mathbf{u}^{(1)}$ should be the solution to the following boundary value problem

$$\left.\begin{aligned} \sigma_{ji,j}^{(1)} &= 0 \\ 2\varepsilon_{ij}^{(1)} &= u_{i,j}^{(1)} + u_{j,i}^{(1)} \\ \sigma_{ij}^{(1)} &= C_{ijkl}\varepsilon_{kl}^{(1)} \end{aligned}\right\} \quad \text{in } V' \tag{18}$$

with

$$\sigma_{ij}^{(1)} = 0 \qquad \text{at infinity} \tag{19}$$

$$\sigma_{yy}^{(1)}(x,0) = 0 \qquad \text{on the open portions of the crack} \tag{20}$$

$$\begin{aligned} \sigma_{xy}^{(1)}(x,0) = {} & \mu\sigma_{yy}^{(1)}(x,0) + \lambda(x)\frac{\mathrm{d}}{\mathrm{d}x}[\sigma_{xx}^{(0)}(x,0) - \mu\sigma_{xy}^{(0)}(x,0)] \\ & - \lambda'(x)[(1+2\mu^2)\sigma_{yy}^{(0)}(x,0) - \sigma_{xx}^{(0)}(x,0)] \qquad \text{for } |x| < a \end{aligned} \tag{21}$$

$$\begin{aligned} & u_y^{(1)}(x,0^+) - u_y^{(1)}(x,0^-) = \\ & \quad -\lambda(x)[\varepsilon_{yy}^{(0)}(x,0^+) - \varepsilon_{yy}^{(0)}(x,0^-)] + \lambda'(x)[u_x^{(0)}(x,0^+) - u_x^{(0)}(x,0^-)] \end{aligned} \tag{22}$$

on the closed sliding portion of the crack.

3. FORMULAE FOR THE STRESS INTENSITY FACTORS

Following Cotterell and Rice[7], let ω be the angle of the crack tip at $x = a$, given by $\omega = \lambda'(a)$ to first order. The normal ($\sigma_{\omega\omega}$) and shear ($\sigma_{r\omega}$) stresses acting along the prolongation of the crack at a small distance r from the tip at $x = a$ are obtained by setting

$\lambda = \omega r + O(\varepsilon^3) = \omega(x-a) + O(\varepsilon^3)$ into eqns (15) and (16). So

$$\sigma_{\omega\omega} = \sigma_{yy}^{(0)}(x,0) - \omega(x-a)\frac{\partial\sigma_{xy}^{(0)}(x,0)}{\partial x} - 2\omega\sigma_{xy}^{(0)}(x,0) + \sigma_{yy}^{(1)}(x,0) + O(\varepsilon^2)$$

$$\sigma_{r\omega} = \sigma_{xy}^{(0)}(x,0) - \omega(x-a)\frac{\partial\sigma_{xx}^{(0)}(x,0)}{\partial x} + \omega[\sigma_{yy}^{(0)}(x,0) - \sigma_{xx}^{(0)}(x,0)] + \sigma_{xy}^{(1)}(x,0) + O(\varepsilon^2).$$

In general, partially-closed crack problems are contact problems and part of the solution is the finding of the closed portions of the crack; the solution depends on crack surface conditions and is often obtained by iteration. For the rest of the paper, we assume that the crack tip is open, so that the stress field has square-root singularities at $x = \pm a$. This is also known to be true if the crack faces are in frictionless contact (Comninou[15, 16], Comninou and Schmueser[17]). Then, the stress intensity factors can be calculated as

$$K_{\text{I}} = \lim_{r\to 0^+} (\sqrt{(2\pi r)}\sigma_{\omega\omega}) = K_{\text{I}}^{(0)} + K_{\text{I}\omega}^{(1)} + K_{\text{I}}^{(1)} + O(\varepsilon^2) \tag{23}$$

$$K_{\text{II}} = \lim_{r\to 0^+} (\sqrt{(2\pi r)}\sigma_{r\omega}) = K_{\text{II}}^{(0)} + K_{\text{II}\omega}^{(1)} + K_{\text{II}}^{(1)} + O(\varepsilon^2) \tag{24}$$

where $K_{\text{I}}^{(0)}$, $K_{\text{II}}^{(0)}$, $K_{\text{I}}^{(1)}$ and $K_{\text{II}}^{(1)}$ are the stress intensity factors for the zero- (eqns (5)–(8)) and first-order (eqns (18)–(22)) problems, and

$$K_{\text{I}\omega}^{(1)} = -\omega\sqrt{(2\pi)} \lim_{x\to a^+}\left[(x-a)^{3/2}\frac{\partial\sigma_{xy}^{(0)}(x,0)}{\partial x} + 2(x-a)^{1/2}\sigma_{xy}^{(0)}(x,0)\right]$$

$$K_{\text{II}\omega}^{(1)} = -\omega\sqrt{(2\pi)} \lim_{x\to a^+}\left\{(x-a)^{3/2}\frac{\partial\sigma_{xx}^{(0)}(x,0)}{\partial x} + (x-a)^{1/2}[\sigma_{yy}^{(0)}(x,0) - \sigma_{xx}^{(0)}(x,0)]\right\}.$$

Using the last two equations and a Williams[11] expansion for the near crack tip stress field, we can show that

$$K_{\text{I}\omega}^{(1)} = -\tfrac{3}{2}\omega K_{\text{II}}^{(0)} \tag{25}$$

and

$$K_{\text{II}\omega}^{(1)} = \tfrac{1}{2}\omega K_{\text{I}}^{(0)}. \tag{26}$$

From the formulation of the first-order problem (eqns (18)–(22)), it is clear that this can be considered as the superposition of the following two problems; problem (i) with a prescribed normal displacement and zero shear traction on the sliding portions of the crack and with the rest of the crack traction free, and problem (ii) with a prescribed shear traction and zero normal traction everywhere on the crack face. Assuming that the sliding and open portions of the crack are known and using the solution of the zero-order problem, we can determine $K_{\text{I}}^{(1)}$ from the solution of problem (i) mentioned above.

As far as $K_{\text{II}}^{(1)}$ is concerned, it is obvious that only the prescribed shear tractions at the crack surface of problem (ii) mentioned above that have opposite directions on the upper and lower surfaces of the crack have a non-zero contribution to $K_{\text{II}}^{(1)}$. With the definition

$$\bar{A}(x) = \tfrac{1}{2}[A(x,0^+) + A(x,0^-)]$$

$K_{\text{II}}^{(1)}$ is known (e.g. Ref. [12]) to be

$$K_{\text{II}}^{(1)} = -\frac{1}{\sqrt{(\pi a)}}\int_{-a}^{a} \bar{\sigma}_{xy}^{(1)}(x)\sqrt{\left(\frac{a+x}{a-x}\right)}\,\mathrm{d}x \tag{27}$$

where, according to (21)

$$\bar{\sigma}_{xy}^{(1)}(x) = \mu\bar{\sigma}_{yy}^{(1)}(x) + \lambda(x)\frac{\mathrm{d}}{\mathrm{d}x}\left[\bar{\sigma}_{xx}^{(0)}(x) - \mu\bar{\sigma}_{xy}^{(0)}(x)\right] - \lambda'(x)\left[(1+2\mu^2)\bar{\sigma}_{yy}^{(0)}(x) - \bar{\sigma}_{xx}^{(0)}(x)\right]. \tag{28}$$

On the other hand, it is possible that the stress field of the zero-order problem, $\boldsymbol{\sigma}^{(0)}$, has the characteristic $1/\sqrt{r}$ elastic singularity at several points in the interval $|x| \leqslant a$; since derivatives of $\boldsymbol{\sigma}^{(0)}$ with respect to x are involved in the formula for $\bar{\sigma}_{xy}^{(1)}$ (eqn (28)), non-integrable singularities will appear in eqn (27). To overcome this difficulty, we assume, for the moment, that the stress components $\sigma_{ij}^{(0)}(x,0)$ are all bounded and differentiable with respect to x in the interval $|x| \leqslant a$; this makes $\sigma_{xy}^{(1)}(x,0)$ also bounded on the crack face. In the case where $\sigma_{ij}^{(0)}(x,0)$ are singular at some point in the interval $|x| \leqslant a$, the singularities are removed by replacing $\sigma_{ij}^{(0)}(x,0)$ by bounded functions that reduce continuously to zero (or any other value that makes $\sigma_{ij}^{(0)}(x,0)$ continuous) over distances closer than a small distance δ to the point where the singularities appear. Later it is shown that it is possible to let δ tend to zero, i.e. effectively to remove the restriction of bounded and differentiable $\sigma_{ij}^{(0)}(x,0)$.

We return now to the calculation of $K_{\mathrm{II}}^{(1)}$. With the above continuity assumptions on $\sigma_{ij}^{(0)}$ we can integrate by parts eqn (28) to find

$$K_{\mathrm{II}}^{(1)} = -\frac{1}{\sqrt{(\pi a)}}\int_{-a}^{a}\{\mu\bar{\sigma}_{yy}^{(1)} - \lambda'[(1+2\mu^2)\bar{\sigma}_{yy}^{(0)} - \bar{\sigma}_{xx}^{(0)}] + [-\lambda' + \tfrac{1}{2}\lambda'(a)](\bar{\sigma}_{xx}^{(0)} - \mu\bar{\sigma}_{xy}^{(0)})\}\sqrt{\left(\frac{a+x}{a-x}\right)}\mathrm{d}x - \frac{1}{\sqrt{(\pi a)}}\int_{-a}^{a}(\bar{\sigma}_{xx}^{(0)} - \mu\bar{\sigma}_{xy}^{(0)})\left[\tfrac{1}{2}\lambda'(a) - \frac{a\lambda + (a-x)\lambda'(a)}{(a-x)^2}\right]\sqrt{\left(\frac{a-x}{a+x}\right)}\mathrm{d}x. \tag{29}$$

It should be noted that $a\lambda(x) + (a-x)\lambda'(a) = (\mathrm{d}/\mathrm{d}x)[a\lambda(x) + (a-x)\lambda'(a)] = 0$ at $x = a$, so there is no divergence at the upper limit of the second integral in (29). It can also be seen that an integrable singularity can exist in $\bar{\sigma}_{ij}^{(0)}(x)$, provided it is not at $x = \pm a$, as was also noted in Ref. [7]. Specifically, in terms of our earlier discussion, δ can be shrunk to zero and in that limit, the result of eqn (29) for $K_{\mathrm{II}}^{(1)}$ approaches the result obtained by inserting directly into eqn (29) the singular, actual $\bar{\sigma}_{ij}^{(0)}(x)$. Such considerations, based essentially on the fact that the final result of eqn (29) for $K_{\mathrm{II}}^{(1)}$ contains $\bar{\sigma}_{ij}^{(0)}(x)$ only (and not derivatives with respect to x), allow us to conclude that eqn (29) is valid for all integrable $\bar{\sigma}_{ij}^{(0)}(x)$ (i.e. not necessarily bounded or continuous).

We mention again that part of the solution to our problem is the finding of the closed portions of the crack and this is often obtained by iteration. Assuming that the contact regions are known and having solved the zero-order problem, we can proceed to solve the first-order problem and use the formulae given in this section to find first-order corrections to the stress intensity factors. In some cases (see Section 4) physical reasoning can be used to determine the contact regions; however, this is not always possible and one must always check whether the obtained solution is consistent with the assumed contact regions.

4. THE PROBLEM OF THE KINKED CRACK

A particular case of the curved crack is the kinked crack shown in Fig. 2. The shape of the kinked crack is given by

$$\lambda(x) = \begin{cases} \dfrac{mb}{b-a}(x+a) & \text{for } -a \leqslant x \leqslant -b \\ mx & \text{for } |x| \leqslant b \\ \dfrac{mb}{b-a}(x-a) & \text{for } b \leqslant x \leqslant a. \end{cases}$$

In this case, $\omega = \lambda'(a) + O(\varepsilon^3) = mb/(b-a) + O(\varepsilon^3)$.

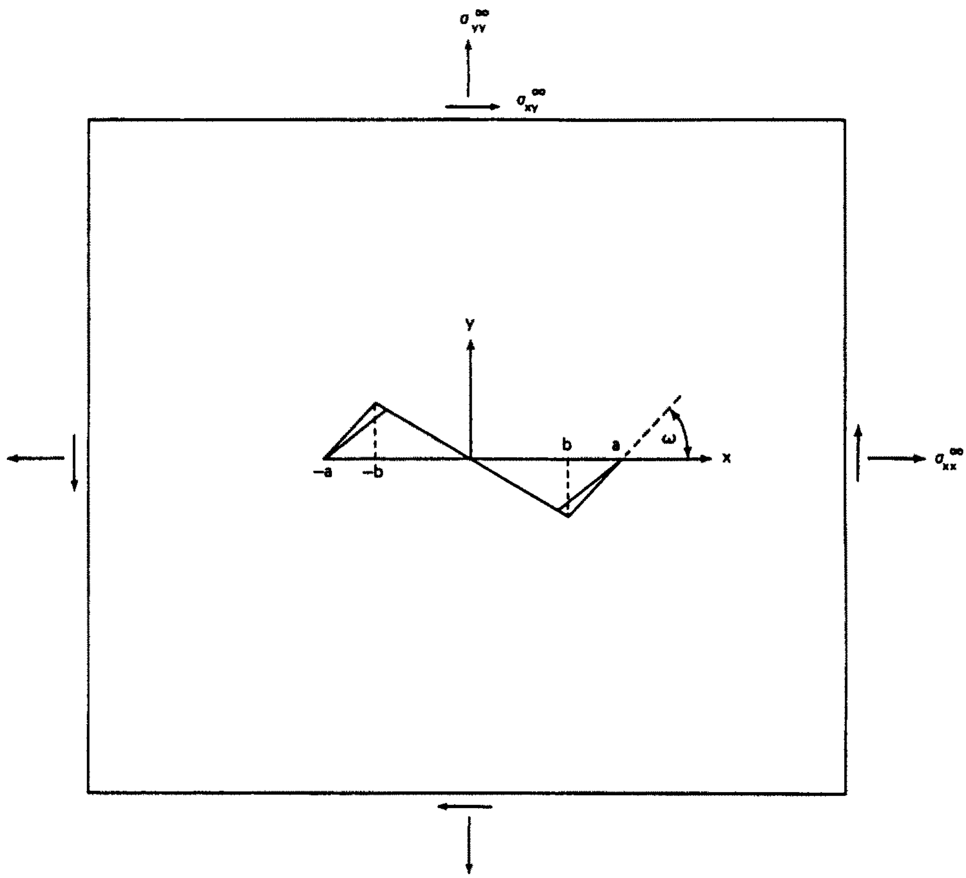

Fig. 2. Infinite plate containing a kinked crack.

Following our previous discussion, we assume that both m and $mb/(b-a)$ are $O(\varepsilon)$, which is equivalent to assuming that $\lambda'(x)$ is $O(\varepsilon)$. We mention again that we are concerned with the case where both σ_{yy}^{∞} and σ_{xy}^{∞} are negative and so, we assume that the portion of the crack in the interval $|x| \leqslant b$ remains closed during the application of the load. Thus, the sliding portion of the crack is the interval $|x| \leqslant b$ and the open portions are the intervals $b \leqslant |x| \leqslant a$. Physically, this is a reasonable assumption to make and it simplifies the whole iterative process of finding the contact regions and their sizes. As mentioned in the previous section, in this paper we are only concerned with the case in which the crack tip remains open and formulae (23)–(26) are applicable. However, there is a possibility that the applied stresses σ_{ij}^{∞} and the orientation of the kinked crack are such that the whole crack remains closed and does not slide. So, after solving the problem assuming that the crack opens in the intervals $b \leqslant |x| \leqslant a$, we must check the validity of this assumption. A necessary condition for this assumption to be true is that K_{I} at the tips of the kinks is positive; if the calculated K_{I} is negative, the tips of the kinks remain closed and the assumption that the crack opens in the intervals $b \leqslant |x| \leqslant a$ is in error.

4.1. *Solution of the zero-order problem*

The zero-order problem can be considered as the superposition of the four problems shown in Fig. 3, where $F(x)$ is the distribution of the $\sigma_{yy}^{(0)}(x, 0)$ stress component of problem 1. We note that for problem 4 the shear stress on the crack face, $\sigma_{xy}^{(0)}(x, 0) = \mu F(x)$, opposes the relative sliding of the crack faces. The quantities of interest for each of the four problems mentioned above are given in the following. In the solutions presented in the rest of this section, conditions of plane strain are assumed; in order to get the plane stress solutions we simply replace ν by $\nu/(1+\nu)$.

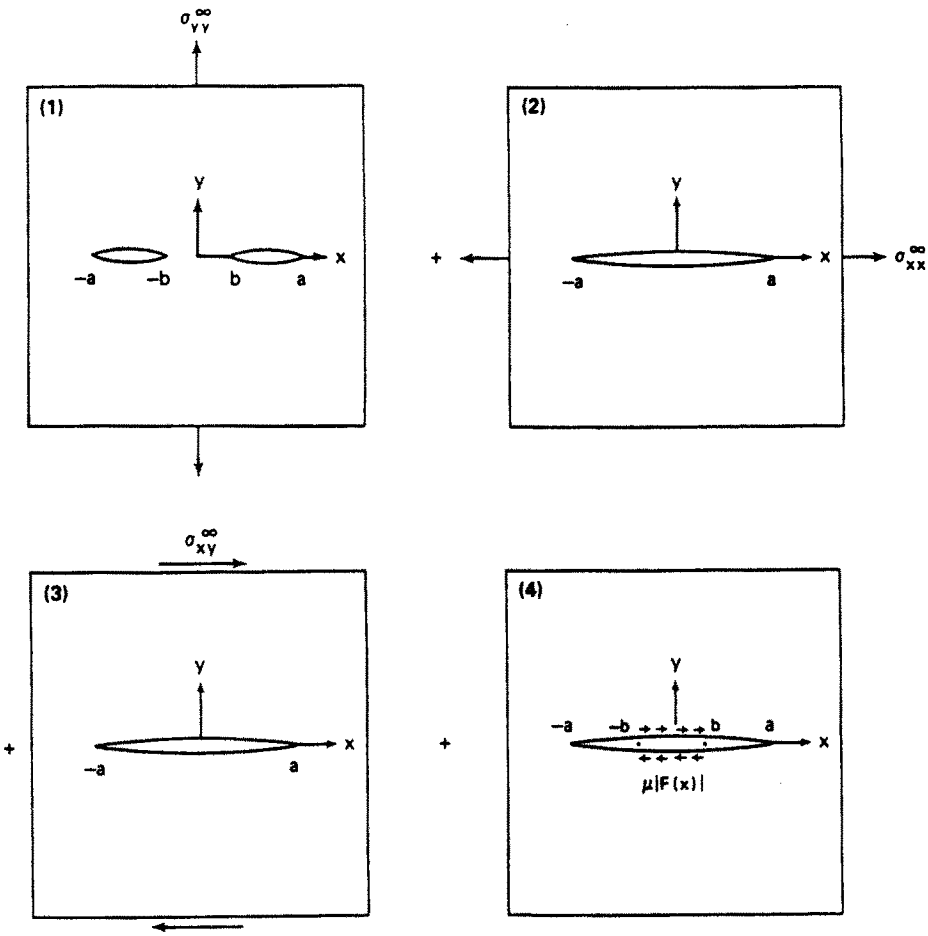

Fig. 3. Superposition used in the solution of the zero-order problem.

4.1.1. *Problem 1*. The solution to this problem has been given by Erdogan[13] and is as follows

$$\sigma_{yy}^{(0)}(x,0) = \frac{\sigma_{yy}^{\infty}}{\sqrt{((b^2-x^2)(a^2-x^2))}}\left[a^2\frac{E(k)}{K(k)}-x^2\right] = F(x) \tag{30}$$

$$\sigma_{xx}^{(0)}(x,0) = F(x)-\sigma_{yy}^{\infty}$$

$$\sigma_{xy}^{(0)}(x,0) = 0$$

$$u_x^{(0)}(x,0^+) = u_x^{(0)}(x,0^-)$$

$$\varepsilon_{yy}^{(0)}(x,0^+) = \varepsilon_{yy}^{(0)}(x,0^-) \qquad \text{for } |x| < b;$$

and

$$\sigma_{xx}^{(0)}(x,0) = -\sigma_{yy}^{\infty} \qquad \text{for } b < |x| < a$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind, respectively, and $k = \sqrt{(1-b^2/a^2)}$.

Also, the stress intensity factors for this problem are

$$K_{\rm I}^{(0)} = \frac{\sigma_{yy}^{\infty}\sqrt{(\pi a)}}{k}\left[1-\frac{E(k)}{K(k)}\right]$$

and

$$K_{\rm II}^{(0)} = 0.$$

Problem 2 consists of a plane strain tension. The solution to this problem is quite obvious; therefore we proceed to problem 3.

4.1.2. *Problem 3*. The solution to this problem is known (e.g. [12]) to be

$$\sigma_{xx}^{(0)}(x, 0^{\pm}) = \mp \sigma_{xy}^{\infty} \frac{2x}{\sqrt{(a^2 - x^2)}}$$

$$\sigma_{yy}^{(0)}(x, 0^{\pm}) = \sigma_{xy}^{(0)}(x, 0^{\pm}) = 0$$

$$\varepsilon_{yy}^{(0)}(x, 0^{\pm}) = \pm \nu \frac{\sigma_{xy}^{\infty}}{G} \frac{x}{\sqrt{(a^2 - x^2)}}$$

and

$$u_x^{(0)}(x, 0^{\pm}) = \pm (1 - \nu) \frac{\sigma_{xy}^{\infty}}{G} \sqrt{(a^2 - x^2)} \qquad \text{for } |x| < a$$

where G is the shear modulus of the material.

In addition

$$K_{\rm I}^{(0)} = 0$$

and

$$K_{\rm II}^{(0)} = \sigma_{xy}^{\infty} \sqrt{(\pi a)}.$$

4.1.3. *Problem 4*. The solution to this problem is derived in Appendix 1 and is as follows

$$\sigma_{xx}^{(0)}(x, 0^{\pm}) = \pm \mu \sigma_{yy}^{\infty} \frac{2x}{\sqrt{(a^2 - x^2)}}$$

$$\sigma_{yy}^{(0)}(x, 0) = 0$$

$$\sigma_{xy}^{(0)}(x, 0) = \mu F(x)$$

$$\varepsilon_{yy}^{(0)}(x, 0^{\pm}) = \mp \nu \frac{\mu \sigma_{yy}^{\infty}}{G} \frac{x}{\sqrt{(a^2 - x^2)}}$$

$$u_x^{(0)}(x, 0^{\pm}) = \mp (1 - \nu) \frac{\mu \sigma_{yy}^{\infty}}{G} \sqrt{(a^2 - x^2)} \qquad \text{for } |x| < b;$$

and

$$\sigma_{xx}^{(0)}(x, 0^{\pm}) = \mp 2 \mu \sigma_{yy}^{\infty} \left\{ \frac{1}{\sqrt{((x^2 - b^2)(a^2 - x^2))}} \left[x^2 - a^2 \frac{E(k)}{K(k)} \right] - \frac{x}{\sqrt{(a^2 - x^2)}} \right\}$$

$$\text{for } b < |x| < a.$$

Also

$$K_{\rm I}^{(0)} = 0$$

$$K_{\rm II}^{(0)} = \frac{\mu \sigma_{yy}^{\infty} \sqrt{(\pi a)}}{k} \left[1 - k - \frac{E(k)}{K(k)} \right].$$

4.1.4. *Superposition*. Superimposing the solutions of the four problems shown in Fig. 3, we find the quantities of interest of the solution to the zero-order problem to be

$$\bar{\sigma}_{xx}^{(0)}(x) = F(x) - \sigma_{yy}^{\infty} + \sigma_{xx}^{\infty} \tag{31}$$

$$\bar{\sigma}_{yy}^{(0)} = F(x) \tag{32}$$

$$\bar{\sigma}_{xy}^{(0)}(x) = \mu F(x) \tag{33}$$

$$\varepsilon_{yy}^{(0)}(x,0^+) - \varepsilon_{yy}^{(0)}(x,0^-) = 2v \frac{\sigma_{xy}^{\infty} - \mu\sigma_{yy}^{\infty}}{G} \frac{x}{\sqrt{(a^2-x^2)}} \tag{34}$$

$$u_x^{(0)}(x,0^+) - u_x^{(0)}(x,0^-) = 2(1-v)\frac{\sigma_{xy}^{\infty} - \mu\sigma_{yy}^{\infty}}{G}\sqrt{(a^2-x^2)} \tag{35}$$

for $|x| < b$; and

$$\bar{\sigma}_{xx}^{(0)}(x) = -\sigma_{yy}^{\infty} + \sigma_{xx}^{\infty} \tag{36}$$

$$\bar{\sigma}_{yy}^{(0)}(x) = \bar{\sigma}_{xy}^{(0)}(x) = 0 \tag{37}$$

for $b < |x| < a$.

Also, the stress intensity factors for the zero-order problem are given by

$$K_{\rm I}^{(0)} = \frac{\sigma_{yy}^{\infty}\sqrt{(\pi a)}}{k}\left[1 - \frac{E(k)}{K(k)}\right] \tag{38}$$

$$K_{\rm II}^{(0)} = (\sigma_{xy}^{\infty} - \mu\sigma_{yy}^{\infty})\sqrt{(\pi a)} + \mu K_{\rm I}^{(0)}. \tag{39}$$

4.2. *Stress intensity factors for the first-order problem*

As discussed in Section 3, the first-order problem can be considered to be the superposition of the two problems shown in Fig. 4.

The mode I stress intensity factor $K_{\rm I}^{(1)}$ is determined by solving problem 1 in Fig. 4, which is actually the problem of the opening of a finite crack by a rigid wedge. The general solution to this problem has been given by Markuzon[14]. Taking into account eqn (22) and the solution of the zero-order problem derived in the previous section, we find that, for our particular case, the shape of the wedge, $h(x)$, is given by

$$2h(x) = u_y^{(1)}(x,0^+) - u_y^{(1)}(x,0^-) = 2m\frac{\sigma_{xy}^{\infty} - \mu\sigma_{yy}^{\infty}}{G}\frac{(1-v)a^2 - x^2}{\sqrt{(a^2-x^2)}}, \qquad |x| \leqslant b. \tag{40}$$

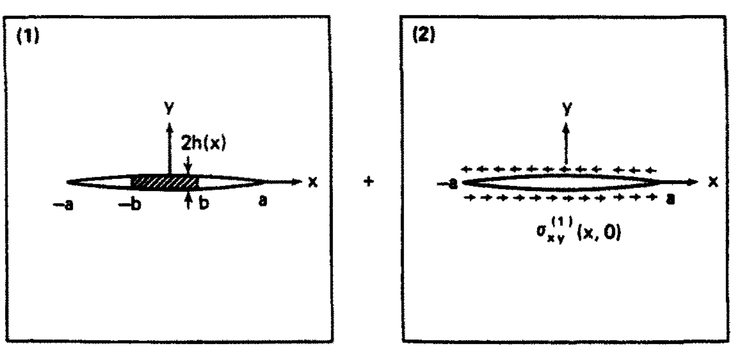

Fig. 4. Superposition used in the solution of the first-order problem.

Using the above formula for the shape of the wedge and Markuzon's[14] solution, we find the mode I stress intensity factor to be (see Appendix 2)

$$K_{\mathrm{I}}^{(1)} = -(\sigma_{xy}^{\infty}-\mu\sigma_{yy}^{\infty})\sqrt{(\pi a)}\frac{m}{1-\nu}\,\frac{k^3-(2-\nu)k^2+(1-2\nu+2C)k+\nu}{2k^2} \tag{41}$$

where

$$C = \frac{1}{K(k)}\left\{\frac{-K(k)k^3+[(1+2\nu)K(k)-2E(k)]k+2\nu}{2k}\right.$$
$$\left.+\nu a\int_b^a \frac{x\sqrt{(x^2-b^2)}-a\sqrt{(a^2-b^2)}}{a^2-x^2}\,\frac{\mathrm{d}x}{\sqrt{((a^2-x^2)(x^2-b^2))}}\right\}.$$

We proceed now to the calculation of the mode II stress intensity factor for the first-order problem, $K_{\mathrm{II}}^{(1)}$, which can be determined either by solving problem 2 in Fig. 4, or equivalently, using eqn (29). We note that $\bar{\sigma}_{xx}^{(0)}(x)$, $\bar{\sigma}_{yy}^{(0)}(x)$ and $\bar{\sigma}_{xy}^{(0)}(x)$ all have the characteristic $1/\sqrt{r}$ singularity at $x = \pm b$; but as discussed in Section 3, eqn (29) for $K_{\mathrm{II}}^{(1)}$ can still be used, provided that the singularities are integrable, which is indeed the case. So, using the solution of the zero-order problem derived in the previous section and applying eqn (29), after some lengthy, but straightforward, integrations we find

$$K_{\mathrm{II}}^{(1)} = \mu K_{\mathrm{I}}^{(1)}+(\sigma_{yy}^{\infty}-\sigma_{xx}^{\infty})\sqrt{(\pi a)}\frac{mb}{b-a}\left(1-\frac{2a}{\pi b}\sin^{-1}\frac{b}{a}\right)+2\mu^2 m\sigma_{yy}^{\infty}\sqrt{(\pi a)}$$
$$+\sigma_{yy}^{\infty}\sqrt{(\pi a)}\frac{m}{2k}\left\{1-5\mu^2+(1+3\mu^2)\frac{E(k)}{K(k)}-\frac{1-\mu^2}{k^2}\left[1-\frac{E(k)}{K(k)}\right]\right\} \tag{42}$$

where $K_{\mathrm{I}}^{(1)}$ is given in eqn (40).

5. DISCUSSION AND COMPARISON WITH THE EXACT SOLUTION

The obtained asymptotic solution for the stress intensity factors at the tips of a kinked crack is of the form

$$K_{\mathrm{I}} = K_{\mathrm{I}}^{(0)}-\tfrac{3}{2}\omega K_{\mathrm{II}}^{(0)}+K_{\mathrm{I}}^{(1)}+O(\varepsilon^2)$$

$$K_{\mathrm{II}} = K_{\mathrm{II}}^{(0)}+\tfrac{1}{2}\omega K_{\mathrm{I}}^{(0)}+K_{\mathrm{II}}^{(1)}+O(\varepsilon^2)$$

where $K_{\mathrm{I}}^{(0)}$, $K_{\mathrm{II}}^{(0)}$, $K_{\mathrm{I}}^{(1)}$ and $K_{\mathrm{II}}^{(1)}$ are given in eqns (38), (39), (41) and (42), respectively. Equations (41) and (42) show that $K_{\mathrm{I}}^{(1)}$ and $K_{\mathrm{II}}^{(1)}$ depend on Poisson's ratio. This is not surprising since displacement boundary conditions have been used along the closed portion of the crack. However, numerical calculations of $K_{\mathrm{I}}^{(1)}$ and $K_{\mathrm{II}}^{(1)}$ show that their dependence on ν is very weak and that their values for plane strain and plane stress are practically indistinguishable. On the other hand, since the zero-order problem is a traction boundary value problem with zero body forces, the leading terms, $K_{\mathrm{I}}^{(0)}$ and $K_{\mathrm{II}}^{(0)}$, are the same under plane strain or plane stress conditions and independent of the elastic constants.

Next, we apply our results to the problem of an infinite plate containing a kinked crack oriented at 36° to the overall compression (Fig. 5). The exact solution, given in Ref. [3], and the asymptotic results for K_{I} are plotted in Fig. 6 vs the angle between the straight crack and its out-of-plane kinks, θ, for several values of the ratio of the length of the kink, l, to the length of the straight crack, c. In general, the region of accuracy of the asymptotic solution depends on both l/c and θ, because the values of m and ω in our analysis, which must be small for the asymptotic solution to be valid, depend on both l/c and θ. But, roughly speaking, the asymptotic solution is seen to be accurate for values of θ up to about 20°. Unfortunately, the

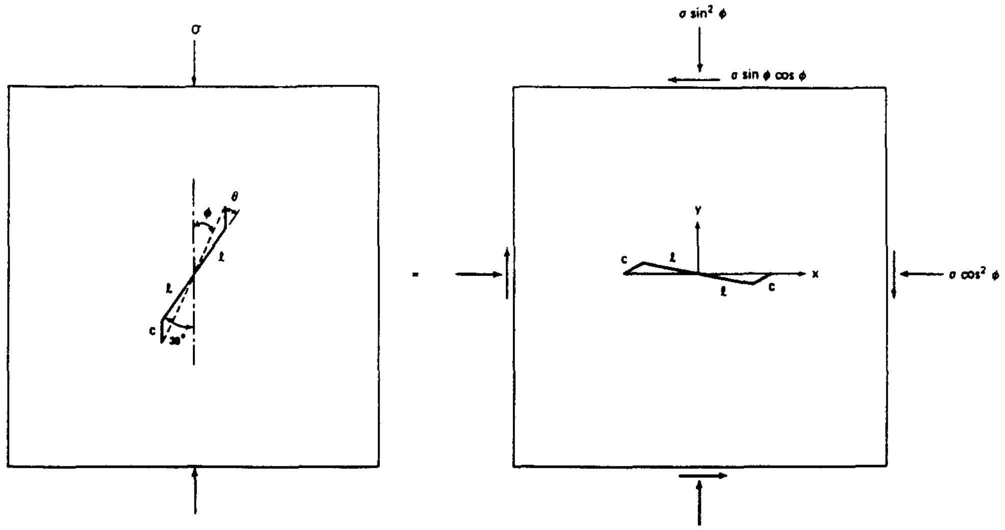

Fig. 5. Infinite plate containing a kinked crack oriented at 36° to the overall compression.

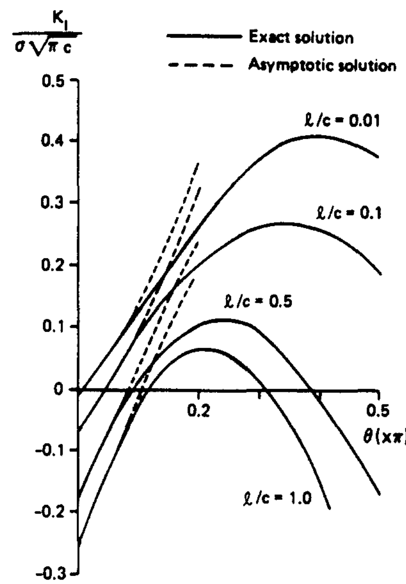

Fig. 6. Stress intensity factor at the tips of the kinked crack shown in Fig. 5 ($\mu = 0.3$).

values of K_{II} for small values of θ are not given in Ref. [3]; so, comparisons of the asymptotic result for K_{II} with the exact solution were not possible.

We mention also that the sign of K_I depends on the orientation of the main crack to the loading direction as well as on l, c, and θ. For some of the cases shown in Fig. 6, K_I is negative. This means that the assumption that the crack opens in the intervals $b \leqslant |x| \leqslant a$ is no longer valid. The correct solution to the problem can be obtained by iteration, i.e. one has to repeat the analysis taking into account the contact zones near the crack tips. This comment concerns both the exact and the asymptotic solution, and poses a challenging problem. The solution to this problem is left for future investigation.

6. CLOSURE

A first-order solution has been obtained for the stress intensity factors at the tips of the kinked extension of a sliding crack. The validity of the asymptotic solution is limited to kinked cracks with small deviations from straightness. There are several situations where this deviation is indeed small. As an example, consider the case of glass plate or a rock block containing several small cracks at different orientations. Under the application of a compressive load, the cracks with an angle to the direction of compression, γ, greater than $\gamma_c = \tan^{-1}(1/\mu)$ will remain closed and only those with $\gamma < \gamma_c$ can, possibly, slide and propagate. It is also known[1–4] that these cracks tend to propagate towards the direction of compression. So, if the coefficient of friction, μ, is very high (which makes γ_c small) the crack propagation will create kinked cracks with small deviations from straightness. For situations like these, the asymptotic results can be used to determine the stress intensity factors and to make predictions for the direction of further propagation. In addition, fatigue due to non-proportional loads can cause the development of cracks that are not straight and are partially closed, although open at the tip. For cases where the deviation from the straight line is small, the methods devised here can be used, although a criterion for determining where the closed portions lie would have to be developed.

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APPENDIX 1

Problem 4 in Fig. 3 is formulated in terms of the complex potentials ϕ and ψ of Muskhelishvili[10]. The stresses and displacements can be expressed as

$$\sigma_{xx}+\sigma_{yy} = 2[\phi'(z)+\overline{\phi'(z)}] \tag{43}$$

$$\sigma_{yy}-\sigma_{xx}+2i\sigma_{xy} = 2[\bar{z}\phi''(z)+\psi'(z)] \tag{44}$$

$$2G(u_x+iu_y) = \kappa\phi(z)-z\overline{\phi'(z)}-\overline{\psi(z)} \tag{45}$$

where $\kappa = 3-4\nu$ for plane strain and $\kappa = (3-\nu)/(1+\nu)$ for plane stress, the overbar denotes the complex conjugate and prime stands for differentiation with respect to $z = x+iy$.

Introducing the analytic function

$$\Omega(z) = z\phi'(z)+\psi(z)$$

eqns (43)–(45) can be written as

$$\sigma_{xx}+\sigma_{yy} = 2[\phi'(z)+\overline{\phi'(z)}] \tag{46}$$

$$\sigma_{yy}-\sigma_{xx}+2i\sigma_{xy} = 2[(\bar{z}-z)\phi''(z)-\overline{\Omega(z)}-\phi'(z)] \tag{47}$$

$$2G(u_x+iu_y) = \kappa\phi(z)-(z-\bar{z})\overline{\phi'(z)}-\overline{\Omega(z)}. \tag{48}$$

For a prescribed shear traction $\sigma_{xy}(x,0)$ along the crack face, it is known (e.g. Ref. [12]) that

$$\phi'(z) = \Omega'(z) = -\frac{i}{2\pi\sqrt{(z^2-a^2)}}\int_{-a}^{a}\sigma_{xy}(x,0)\sqrt{(a^2-x^2)}\frac{dx}{x-z}. \tag{49}$$

In our problem

$$\sigma_{xy}(x,0) = \begin{cases}\mu F(x) & \text{for } |x|<b\\ 0 & \text{for } b<|x|<a\end{cases} \tag{50}$$

where $F(x)$ is defined in eqn (30). Substituting eqn (49) into eqn (50) and carrying out the integration we find

$$\phi'(z) = \bar{\Omega}'(z) = -\frac{i}{2}\mu\sigma_{yy}^{\infty}\left\{\left[z^2-a^2\frac{E(k)}{K(k)}\right]\frac{1}{\sqrt{((z^2-a^2)(z^2-b^2))}}-\frac{z}{\sqrt{(z^2-a^2)}}\right\}.$$

Finally, using eqns (46)–(48) and the definition

$$K_I+iK_{II} = \lim_{x\to a^+}\sqrt{(2\pi(x-a))}[\sigma_{yy}(x,0)+i\sigma_{xy}(x,0)] \tag{51}$$

we find the results shown in Section 4.1.3.

APPENDIX 2

The solution to the problem of the opening of a finite crack by a rigid wedge has been given by Markuzon[14]. In terms of Muskhelishvili's[10] complex potentials, the solution is shown to be

$$\phi'(z) = \bar{\Omega}'(z) = \frac{2G}{\pi(\kappa+1)}\frac{1}{X(z)}\int_{-b}^{b}\frac{dh}{dx}X(x)\frac{dx}{x-z}+\frac{C_0}{X(z)} \tag{52}$$

where $h(x)$ is the function determining the shape of the wedge of length $2b$ (see Fig. 4), $X(z)=\sqrt{((a^2-z^2)(b^2-z^2))}$ and the constant C_0 is determined from the equation

$$\frac{1}{\pi}\int_{b}^{a}\frac{1}{\sqrt{((a^2-x^2)(x^2-b^2))}}\left[\int_{-b}^{b}\frac{dh}{dt}\sqrt{((a^2-t^2)(b^2-t^2))}\frac{dt}{t-x}\right]dx-\frac{\kappa+1}{2G}\frac{K(k)}{a}C_0 = -h(b).$$

As discussed in Section 4.2, the shape of the wedge for our problem is given by

$$h(x) = m\frac{\sigma_{xy}^{\infty}-\mu\sigma_{yy}^{\infty}}{G}\frac{(1-\nu)a^2-x^2}{\sqrt{(a^2-x^2)}}, \qquad |x|<b. \tag{53}$$

Substituting eqn (53) into eqn (52) and carrying out the integrations, we find

$$\phi'(z) = \bar{\Omega}'(z) = -\frac{ma^2}{2(1-\nu)}\frac{\sigma_{xy}^{\infty}-\mu\sigma_{yy}^{\infty}}{\sqrt{((z^2-a^2)(z^2-b^2))}}$$
$$\times\left[\nu\frac{z\sqrt{(z^2-b^2)}-a\sqrt{(a^2-b^2)}}{z^2-a^2}+\frac{z^2-z\sqrt{(z^2-b^2)}}{a^2}-\frac{2\nu a^2+b^2}{2a^2}+C\right]$$

where

$$C = \frac{1}{K(k)}\left\{\frac{-K(k)k^3+[(1+2\nu)K(k)-2E(k)]k+2\nu}{2k}\right.$$
$$\left.+\nu\int_{b}^{a}\frac{x\sqrt{(x^2-b^2)}-a\sqrt{(a^2-b^2)}}{a^2-x^2}\frac{dx}{\sqrt{((a^2-x^2)(x^2-b^2))}}\right\}.$$

Finally, using eqns (46), (47) and (51) we find the stress intensity factor to be

$$K_I = -(\sigma_{xy}^{\infty}-\mu\sigma_{yy}^{\infty})\sqrt{(\pi a)}\frac{m}{1-\nu}\frac{k^3-(2-\nu)k^2+(1-2\nu+2C)k+\nu}{2k^2}.$$